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On Weakly Sign-Symmetric Matrices

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ABSTRACT

Weakly sign-symmetric matrices have non-negative principal minors and non-negative products of symmetrically placed pairs of almost-principal minors. A necessary condition is proved for such a matrix to have as rank a given positive integer. Several characterizations are given of those weakly sign-symmetric matrices for which the generalized Hadamard inequality holds.

INTRODUCTION

We use terminology and notation similar to that of Carlson [1], [2], and Marcus and Minc [10]. Let α, β and $\gamma, \ldots$ be subsets of $Z_n = \{1, 2, \ldots, n\}$ and assumed to be in their natural order. Let $|\alpha|$ be the order of α. Let A be an $n \times n$ complex matrix; then $A[\alpha|\beta]$ is the submatrix of A with rows and columns indexed by α and β, respectively, and if $|\alpha| = |\beta|$, $A(\alpha|\beta) = \det A[\alpha|\beta]$. We let $A[\alpha] = A[\alpha|\alpha]$ and $A(\alpha) = A(\alpha|\alpha)$; if α is empty then $A(\alpha) = 1$. Next for convenience let $\hat{\imath}$ be the complement of i in Z_n, let $\alpha - i = \alpha - \{i\}$, and let $\{\alpha, i\} = \alpha \cup \{i\}$.

The term A is a weakly sign-symmetric matrix or WSS-matrix if and only if $A(\alpha) \geqslant 0$ for all $\alpha \subseteq Z_n$ and

$$A(\alpha, i|\beta, j)A(\alpha, j|\alpha, i) \geqslant 0 \quad \text{for all} \quad \alpha \subseteq Z_n, \quad i, j \in Z_n - \alpha. \tag{1}$$

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A minor of the type $A(\alpha, i|\alpha, j)$, $\alpha \subseteq Z_n$, $i, j \in Z_n - \alpha$, will be called almost principal.

The term A is a generalized Hadamard or GH-matrix if and only if $A(\alpha) \geqslant 0$ for all $\alpha \subseteq Z_n$ and

$$A(\alpha \cup \beta)A(\alpha \cap \beta) \leqslant A(\alpha)A(\beta) \quad \text{for all} \quad \alpha, \beta \subseteq Z_n. \tag{2}$$

The classical inequalities of Hadamard

$$a_{11}a_{22}\cdots a_{nn} \geqslant \det A,$$

and Fischer,

$$A(\alpha \cup \beta) \leqslant A(\alpha)A(\beta) \quad \text{for all} \quad \alpha, \beta \subseteq Z_n, \quad \alpha \cap \beta = \phi,$$

are special cases of (2). For related results see Carlson [2] and Fan [3], [4].

It is obvious that positive semi-definite and totally non-negative matrices are WSS, and it is easy to show that M-matrices (which we allow to have non-negative principal minors) are also WSS. It is also true that all matrices in these classes are GH. However, not all WSS-matrices are GH; the most obvious examples being the $k \times k$ cyclic permutation matrices with $k \geqslant 3$ and k odd.

THEOREM 1. *Every GH-matrix is a WSS-matrix. The WSS-matrix A is a GH-matrix if and only if*

$$\textit{for} \quad \alpha \subseteq \beta \subseteq Z_n, \quad A(\beta) = 0 \quad \textit{whenever} \quad A(\alpha) = 0. \tag{3}$$

Proof. Gantmacher and Krein [6, p. 111] and Carlson [1] have shown that if all principal minors of A are positive, then A is GH if and only if it is WSS. Carlson's proof that GH-matrices are WSS actually holds in general. To prove our second statement, suppose first that A is GH. If $A(\alpha) = 0$ and $\alpha \subseteq \beta$, then

$$0 \leqslant A(\beta) \leqslant A(\alpha)A(\beta - \alpha) = 0,$$

and $A(\beta) = 0$. Conversely, suppose A is WSS and that (3) holds. Given $\alpha, \beta \subseteq Z_n$, either $A(\alpha \cup \beta) = 0$ (in which case $A(\alpha \cup \beta)A(\alpha \cap \beta) \leqslant A(\alpha)A(\beta)$, since $A(\alpha) \geqslant 0$, $A(\beta) \geqslant 0$) or $A(\alpha \cup \beta) > 0$ (in which case all principal minors of $A[\alpha \cup \beta]$ are positive, by (3), and Carlson's proof of (2) holds.

THE RANK OF A WSS-MATRIX

We next study the relationship between nested principal minors of general WSS-matrices. Let $\rho(A)$ denote the rank of A.

LEMMA 1. *Let A be an $n \times n$ WSS-matrix. Suppose $A(\alpha) > 0$ for some $\alpha \subseteq Z_n$, $0 \leqslant |\alpha| \leqslant n-2$, and $A(\alpha, s) = 0$ for some $s \in Z_n - \alpha$. For any $t \in Z_n - \{\alpha, s\}$, let $\gamma = \{\alpha, s, t\}$; then*

$$A(\gamma) = A(\alpha, s, t) = 0, \tag{4}$$

and

$$A(\alpha, s|\alpha, t) A(\alpha, t|\alpha, s) = 0. \tag{5}$$

Further, if $A(\alpha, s|\alpha, t) = 0$, then

$$A(\alpha, s|\gamma - r) = 0 \quad \text{for all} \quad r \in \gamma. \tag{6}$$

If $A(\alpha, t|\alpha, s) = 0$ then

$$A(\gamma - r|\alpha, s) = 0 \quad \text{for all} \quad r \in \gamma. \tag{6'}$$

Proof. Let $d_{ij} = A(\alpha, i|\alpha, j)$ for $i, j \in \{s, t\}$. The hypotheses, together with Sylvester's identity (cf. [5, Vol. I, p. 33]) imply that

$$0 \leqslant A(\alpha)A(\gamma) = \det(d_{ij}) = -d_{st}d_{ts} = -A(\alpha, s|\alpha, t)A(\alpha, t|\alpha, s). \tag{7}$$

Since A is WSS, we must have equality in (7), which is (5); also, since $A(\alpha) > 0$, we have (4).

If $\alpha = \phi$, (6) and (6′) are obvious. Assume then that $\alpha \neq \phi$. Suppose $A(\alpha, s|\alpha, t) = 0$. By hypothesis, since $A(\alpha) > 0$, the set C of columns of $A[\alpha, s|\alpha]$ is linearly independent. As $A(\alpha, s) = A(\alpha, s|\alpha, t) = 0$, the columns $A[\alpha, s|s]$ and $A[\alpha, s|t]$ are linearly dependent on C. Since all columns of $A[\alpha, s|\gamma]$ are linearly dependent on C, which has only $|\alpha|$ elements, (6) holds. If $A(\alpha, t|\alpha, s) = 0$, a similar argument gives (6′).

Our next result is essentially known for positive semi-definite matrices. Since positive, semi-definite matrices are WSS, this follows from Lemma 1 above and Theorem 6 of [11, p. 92]. For related results for totally non-negative matrices, see Gantmacher and Krein [6, pp. 113–115] or Karlin [8, pp. 89–91].

THEOREM 2. *Let A be an $n \times n$ WSS-matrix. Suppose that for some p, $1 \leqslant p \leqslant n - 2$, that all principal minors of order p are positive, and all principal minors of order $p + 1$ are zero. Then the rank of A is p.*

Proof. We shall show that for every $\beta = \{\alpha, i\}$, $|\alpha| = p$, $i \notin \alpha$, and $j \notin \alpha$, the column $A[\beta|j]$ is linearly dependent on the columns of $A[\beta|\alpha]$. It will follow that for every β', $\beta \neq \beta'$ and $|\beta'| = p + 1$, $A(\beta|\beta') = 0$. From this it will follow that $\rho(A) = p$.

Without loss of generality, we may assume $\alpha = \{1, 2, \ldots, n - 2\}$, $i = n - 1$, $j = n$. Now all principal minors of order $n - 2$ are positive, and those of order $n - 1$ are zero. By Lemma 1, $A(\widehat{n-1}|\hat{n})A(\hat{n}|\widehat{n-1}) = 0$. By repeated application of Lemma 1, if $A(\widehat{n-1}|\hat{n}) = 0$, then for every $k \in \alpha$, $A(\hat{k}|\hat{n}) = 0$, and hence $A(\hat{k}|\widehat{n-1}) = 0$, and hence $A(\hat{n}|\widehat{n-1}) = 0$. Now we have that $A(\hat{n}|\widehat{n-1}) = 0$, so by the proof of Lemma 1, for $j \neq n$, $A[\hat{n}|j]$ is linearly dependent on the columns of $A[\hat{n}|\alpha]$.

COROLLARY. *Let A be an $n \times n$ WSS-matrix. Suppose for some p, $1 \leqslant p < n$, that all principal submatrices of order p are non-singular and generalized Hadamard and that all principal minors of order $p + 1$ are zero. Then A is a GH-matrix.*

Proof. The proof follows from Theorems 1 and 2.

A CHARACTERIZATION OF WSS-MATRICES WHICH ARE NOT GH

In this section we show that every WSS-matrix which is not GH is related to the example, given earlier, of a $k \times k$ cyclic permutation matrix.

LEMMA 2. *Let A be a WSS-matrix with* $\det A > 0$ *and $A[\alpha]$ a GH-matrix for all $\alpha \subsetneq Z_n$. The following are equivalent:*

(a) *A is not a GH-matrix.*

(b) *There exists $\alpha \neq Z_n$ such that $A(\alpha) = 0$.*

(c) Adj A *has a non-singular principal submatrix, of order $k \geqslant 3$, which is a cyclic permutation of a diagonal matrix.*

Proof. Conditions (a) and (b) are equivalent by Theorem 1.

Suppose (b) holds. We will first show that all principal minors of A of orders $n - 2$ and $n - 1$ are zero. By Theorem 1 there exists $\gamma_0 \subseteq Z_n$, with $\alpha \subseteq \gamma_0$ and $|\gamma_0| = n - 1$, such that $A(\gamma_0) = 0$. Let $\gamma \subseteq Z_n$, $|\gamma| = n - 1$, with $\gamma \neq \gamma_0$. Let $\{i_0\} = Z_n - \gamma_0$ and $\{j_0\} = Z_n - \gamma$. Then $\gamma_0 \cap \gamma$ contains $n - 2$ elements of Z_n and $\gamma_0 \cap \gamma \subseteq \gamma_0$. Since $\det A > 0$, all principal minors of order $n - 2$ of $A[\gamma_0]$ are zero by Lemma 1. Therefore $A(\gamma_0 \cap \gamma) = 0$; and since $\gamma_0 \cap \gamma \subset \gamma$ and $A[\gamma]$ is a GH-matrix, $A(\gamma) = 0$. Thus all principal minors of order $n - 1$ of A are zero. It follows from this and $\det A > 0$, by Lemma 1, that all principal minors of order $n - 2$ of A are zero.

Thus by Jacobi's Theorem [5, Vol. I, p. 21] all principal minors of $B = \operatorname{Adj} A$ are non-negative, and all diagonal elements and 2×2 principal minors of B are zero. Let k be the minimum value, $3 \leqslant k \leqslant n$, such that there exists $B[\psi]$ of order k with $B(\psi) > 0$ and $B(\alpha) = 0$ for all $\phi \neq \alpha \subsetneq \psi$. Then by Theorem 2.1.10 of [7], $B[\psi]$ is a cyclic permutation of a non-singular diagonal matrix.

Conversely, suppose (c) holds. Since $B[\psi]$ and therefore $B = \operatorname{Adj} A$ has a zero diagonal element, Jacobi's Theorem implies that A has a zero principal minor of order $n - 1$; and since $\det A > 0$, (b) holds.

For $1 \leqslant m \leqslant n$, let $C_m(A)$ denote the mth compound of A [10, p. 16]. Since by Theorem 1, A is a GH-matrix if and only if all principal submatrices of A are GH-matrices, the following holds.

THEOREM 3. *Let A be a WSS-matrix. A is not a GH-matrix if and only if for some m, $1 \leqslant m \leqslant n$, $C_{m-1}(A)$ has a principal submatrix of order $k \geqslant 3$, which is a cyclic permutation of a non-singular diagonal matrix, and which is determined by precisely m rows of A.*

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REFERENCES

1 D. Carlson, Weakly sign-symmetric matrices and some determinantal inequalities, *Colloq. Math.* 27(1967), 123–129.
2 D. Carlson, On some determinantal inequalities, *Proc. A. M. S.* 19(1968), 462–466.
3 Ky Fan, An inequality for subadditive functions on a distributive lattice, with applications to determinantal inequalities, *Lin. Alg. Appl.* 1(1968), 33–38.
4 Ky Fan, Subadditive functions on a distributive lattice and on extensions of Szasz's inequality, *J. Math. Anal. Appl.* 18(1967), 262–268.
5 F. R. Gantmacher, *The Theory of Matrices*, Chelsea, New York (1960), 2 vols.

6 F. R. Gantmacher and M. G. Krein, *Oszillationsmatrizen, Oszillationskerne und kleine Schwingungen mechanischer Systeme*, Akademie-Verlag, Berlin (1960).

7 B. M. Green, Characterizations of matrices for which certain determinantal equalities hold, Ph.D. thesis, Oregon State University (1969), 75 numb. leaves (microfilm).

8 S. Karlin, *Total Positivity*, Stanford University Press, Stanford (1968).

9 D. M. Koteljanskii, A property of sign-symmetric matrices (Russian), *Ukrain. Mat. Z.* 2(1950); English transl.: Ser. 2, 27(1963), 19–23.

10 M. Marcus and H. Minc, *A Survey of Matrix Theory and Matrix Inequalities*, Allyn and Bacon, Boston (1964).

11 H. W. Turnbull and A. C. Aitken, *An Introduction to the Theory of Canonical Matrices*, Dover, New York (1961).

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